A QUADRATURE OF THE TWO-DIMENSIONAL EQUATIONS OF THE HYDRODYNAMICS OF AN INCOMPRESSIBLE IDEAL LIQUID AND A MODEL OF THE FLOW AROUND CONVEX SYMMETRIC BODIES WITH SUCTION OR INJECTION

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A class of solutions of the steady-state hydrodynamic equations of an incompressible liquid has been considered in the Euler approximation when the pressure field is described by an additive function of the form $P(x; y) = F(x) + G(y)$. It turns out that this class of solutions includes potential flows such as wave motion of a liquid with a finite depth, a flow inside a right angle, and a hypersonic gas flow around a sphere in the approximation of constant density near the stagnation point. Among nonpotential flows, this class includes, in particular, a hypersonic flow around a cylinder. The results obtained are used to construct a model of a flow around convex symmetric bodies with suction or injection. This model can be of interest in solving some problems of physicochemical technology, for example, separation of gas mixtures and isotopes.

1. Consider plane and axisymmetric flows of an incompressible liquid in terms of the Euler steady-state equations of hydrodynamics

$$\frac{\partial (ux^{\nu})}{\partial x} + \frac{\partial (vx^{\nu})}{\partial y} = 0 , \tag{1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x}, \tag{2}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y}, \tag{3}$$

where u and v are the components of the velocity vector $\mathbf{V}$ in the x and y directions; $\nu = 0$ or 1 for plane or axisymmetric flows, respectively. We are interested in the case where the components of the pressure gradient have the form

$$- \frac{1}{\rho} \frac{\partial p}{\partial x} = f(x) ; \quad - \frac{1}{\rho} \frac{\partial p}{\partial y} = g(y) . \tag{4}$$

This means that the pressure field in the liquid is an additive function defined by

$$P(x ; y) = F(x) + G(y) = P(x_0 ; y_0) - \rho \left[\int f(x)\, dx + g(y)\, dy\right], \tag{5}$$

where integration is carried out along the streamline originating at the point (x_0, y_0). Our objective is to determine the class of liquid flows corresponding to the case expressed by Eqs. (4). For this purpose the stream function $\psi(x; y)$ is introduced by means of the relations

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$$u = \frac{1}{x^{\nu}} \frac{\partial \psi}{\partial y}; \quad v = -\frac{1}{x^{\nu}} \frac{\partial \psi}{\partial x} \tag{6}$$

and expressed in the form

$$\psi(x;\ y) = \eta(x)\,\xi(y)\,. \tag{7}$$

After substitution of (4), (6), and (7) into system of equations (1)-(3), the following system of equations is obtained:

$$\eta(\xi')^2 \left(\eta' - \frac{\nu}{x}\eta\right) - \eta\eta'\,\xi\xi'' = x^{2\nu} f(x)\,, \tag{8}$$

$$\eta\,\xi\xi' \left(\frac{\nu}{x}\eta' - \eta''\right) + (\eta')^2\,\xi\xi' = x^{2\nu} g(y)\,, \tag{9}$$

where the prime indicates differentiation with respect to the variable on which the function depends.

2. First, a plane flow will be considered: $\nu = 0$. In this case Eqs. (8) and (9) take the form of the following symmetric system:

$$\eta\eta'\,[(\xi')^2 - \xi\xi''] = f(x)\,, \tag{10}$$

$$\xi\xi'\,[(\eta')^2 - \eta\eta''] = g(y)\,, \tag{11}$$

which is equivalent to the following chain of equations:

$$(\eta')^2 - \eta\eta'' = 1\,; \quad (\xi')^2 - \xi\xi'' = 1\,, \tag{12}$$

$$\eta\eta' = f(x)\,; \quad \xi\xi' = g(y)\,. \tag{13}$$

Integration of Eq. (12) gives

$$\eta(x) = C_2 \exp(C_1 x) - \frac{\exp(-C_1 x)}{4C_1^2 C_2}\,, \tag{14}$$

$$\xi(y) = C_4 \exp(C_3 y) - \frac{\exp(-C_3 y)}{4C_3^2 C_4}\,, \tag{15}$$

where $C_1, \ldots, C_4$ are integration constants determined by the conditions of the problem. After substitution of (14) and (15) into (7) and (13), the stream function is expressed as

$$\psi(x;\ y) = \left[C_2 \exp(C_1 x) - \frac{\exp(-C_1 x)}{4C_1^2 C_2}\right] \left[C_4 \exp(C_3 y) - \frac{\exp(-C_3 y)}{4C_3^2 C_4}\right] \tag{16}$$

and the functions $f(x)$ and $g(y)$ are expressed as

$$f(x) = C_1 C_2^2 \exp(2C_1 x) - \frac{\exp(-2C_1 x)}{16 C_1^3 C_2^2}\,, \tag{17}$$

$$g(y) = C_3 C_4^2 \exp(2C_3 y) - \frac{\exp(-2C_3 y)}{16 C_3^3 C_4^2}. \tag{18}$$

Relations (16)-(18) determine the class of solutions of this problem with limitations (4) imposed on the pressure gradient. We will find what physically reasonable liquid flows correspond to these solutions.

3. To answer this question, first we will try to isolate the class of potential flows of a liquid described by general solutions (16)-(18). As is known, in this case the stream function should satisfy the Laplace equation, which with account for Eq. (7) will assume the form

$$\eta'' \xi + \eta \xi'' = 0. \tag{19}$$

Substitution of solutions (14), (15) into Eq. (19) shows that the condition of a potential flow is satisfied if the constants C_1 and C_3 are connected by the relation

$$C_1^2 + C_3^2 = 0. \tag{20}$$

Omitting the trivial case $C_1 = C_3 = 0$, it can be seen from (20) that the condition for a potential flow is fulfilled if one of the two constants C_1 and C_3 is an imaginary number. Let $C_1 = iC_3 = iC$ for definiteness. Moreover, $C_4 = C_2 = 1/2C_3 = 1/2C$ is assumed. Then expression (16) for the stream function will take the form

$$\psi(x; y) = \frac{1}{C^2} \cos(Cx) \operatorname{sh}(Cy). \tag{21}$$

The corresponding components of the velocity are found using relation (6):

$$u = \frac{1}{C} \cos(Cx) \operatorname{ch}(Cy), \tag{22}$$

$$v = \frac{1}{C} \sin(Cx) \operatorname{sh}(Cy). \tag{23}$$

If $y = z + h$ is assumed, solutions (22) and (23) will be solutions for the wave motion of a liquid with the finite depth h, which were obtained in [1] by direct integration of the wave equation. It is rather difficult to predict this result starting from the hydrodynamic equations, as is done in the present work. It should be noted that solutions (22) and (23) correspond to a coordinate system moving in the direction opposite to the propagation of the wave front in such a way that their relative velocity is zero.

Another example of a potential flow that can be singled out from solution (16) is obtained if $C_2 = 1/2C_1$ and $C_4 = 1/2C_3$ are assumed. Then, the stream function is expressed by

$$\psi(x; y) = \frac{1}{C_1 C_3} \operatorname{sh}(C_1 x) \operatorname{sh}(C_3 y). \tag{24}$$

For small values of the arguments in (24) $\psi(x; y) \approx xy$ is obtained. This stream function satisfies Laplace equation (19) and consequently describes the potential flow known as the flow of a liquid inside a right angle [1].

Consider the stream function of form (24) for small values of the argument x. In this case

$$\psi(x; y) = \frac{x}{C_3} \operatorname{sh}(C_3 y). \tag{25}$$

This stream function does not describe a potential flow but it is interesting for another reason. Using relations (6), it follows from (25) that the components of the flow velocity are expressed as

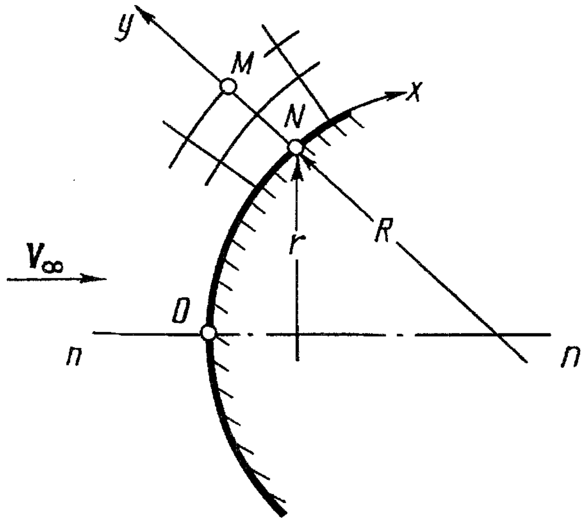

Fig. 1. Coordinate system for a hypersonic gas flow around a body.

$$u = x\,\mathrm{ch}\,(C_3 y)\,; \quad v = -\frac{1}{C_3}\,\mathrm{sh}\,(C_3 y)\,. \tag{26}$$

Expressions (26) correspond to solutions known from the theory of hypersonic gas flow around a cylinder in the approximation of constant density near the stagnation point when the Rankine–Hugoniot relations on the shock wave and conditions on the surface of the body are taken as boundary conditions [2]. In this case the distance Δ from the surface of the body in the flow to the shock wave is determined by the coordinate y at which the flow velocity is equal to the gas velocity behind the shock wave. It should be noted that the coordinates x and y in expressions (25), (26) correspond to a curvilinear coordinate system in which the position of the point M is determined by the distance $y = NM$ along the normal from the body surface and by the arc length $x = ON$ of the contour in the flow with the central point O taken as the reference point (Fig. 1). For this system of coordinates the Euler steady-state equations become Eqs. (1)-(3) in the approximation $x/r \approx 1$, $y/R << 1$ [3].

4. Now the axisymmetric flow $\nu = 1$ will be considered. In this case Eqs. (8) and (9) become

$$\eta\,(\xi')^2\left(\eta' - \frac{1}{x}\eta\right) - \eta\eta'\,\xi\xi'' = x^2 f(x)\,, \tag{27}$$

$$\xi\xi'\left[\frac{1}{x}\eta\eta' - \eta\eta'' + (\eta')^2\right] = x^2 g(y)\,. \tag{28}$$

It can be shown that Eq. (27) is valid only when $\xi(y)$ is a linear function:

$$\xi\,(y) = C_1 y + C_2\,, \tag{29}$$

where C_1 and C_2 are constants determined by the conditions of the problem. With account for (29) we have the following equation for the function $\eta x)$:

$$\eta\eta' - \frac{1}{x}\eta^2 = \frac{x^2}{C_1^2} f(x)\,, \tag{30}$$

the solution of which is

$$\eta\,(x) = \pm\, x\sqrt{\left(\frac{2}{C_1^2}\int f(x)\,dx + C_3\,,\right)}\,, \tag{31}$$

where C_3 is the integration constant. After substitution of (29) and (31) into (7) the stream function is obtained:

$$\psi(x;\, y) = \pm x \sqrt{\left(\frac{2}{C_1^2} \int f(x)\, dx + C_3 (C_1 y + C_2)\right)}\,. \tag{32}$$

The function $g(y)$ is found after substitution of (29) into Eq. (28):

$$g(y) = \xi\xi' = C_1 (C_1 y + C_2)\,. \tag{33}$$

Then, solution (31) should satisfy the equation

$$\frac{1}{x}\eta\eta' - \eta\eta'' + (\eta')^2 = x^2\,, \tag{34}$$

which is used to determine the function $f(x)$. Since Eq. (34) cannot be solved in quadratures, it is impossible to obtain an expression for $f(x)$ in explicit form in this case. Nevertheless, it can be assumed that solutions (32) and (33) with condition (34) give a solution of the problem formulated in implicit form.

5. We will find what physically reasonable flows of a liquid correspond to solutions (32)-(34). As before, for this purpose a class of potential flows described by these solutions will be separated out. In the case of an axisymmetric flow the Laplace equation for the stream function $\psi(x;\, y)$ has the form [1]

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} - \frac{1}{x}\frac{\partial \psi}{\partial x} = 0 \tag{35}$$

or with account for (7) and (29)

$$\xi \left(\eta'' - \frac{1}{x}\eta'\right) = 0\,. \tag{36}$$

Since $\xi(y) \neq 0$, for potential flows the function $\eta(x)$ should be a solution of the bracketed equation in (36):

$$\eta(x) = \frac{C_3' x^2}{2} + C_4'\,. \tag{37}$$

The constants $C_3' = \pm 1$ and $C_4' = 0$ are obtained after substitution of solution (37) into Eq. (34). Then, the stream function of this potential flow takes the form

$$\psi(x;\, y) = \frac{x^2}{2}(C_1 y + C_2)\,, \tag{38}$$

and the function $f(x) = C_1^2 x$ is obtained from comparison of solutions (37) and (31). Using relations (6) with $\nu = 1$, we determine the components of the velocity vector from expression (38) ($C_2 = 0$ is assumed):

$$u = \frac{C_1 x}{2}; \quad v = -C_1 y\,. \tag{39}$$

Relation (39) corresponds exactly to the solution obtained in the theory of a hypersonic gas flow around a sphere in the approximation of constant density near the stagnation point if the curvilinear system shown in Fig. 1 is used as a coordinate system [2]. In this coordinate system the Euler steady-state equations become Eqs. (1)-(3) in the same approximation as was used in Sec. 3. It should be noted that unlike flow around a sphere, under similar conditions a hypersonic flow around a cylinder is a nonpotential flow. This can probably be explained by the fact that an axisymmetric flow is three-dimensional.

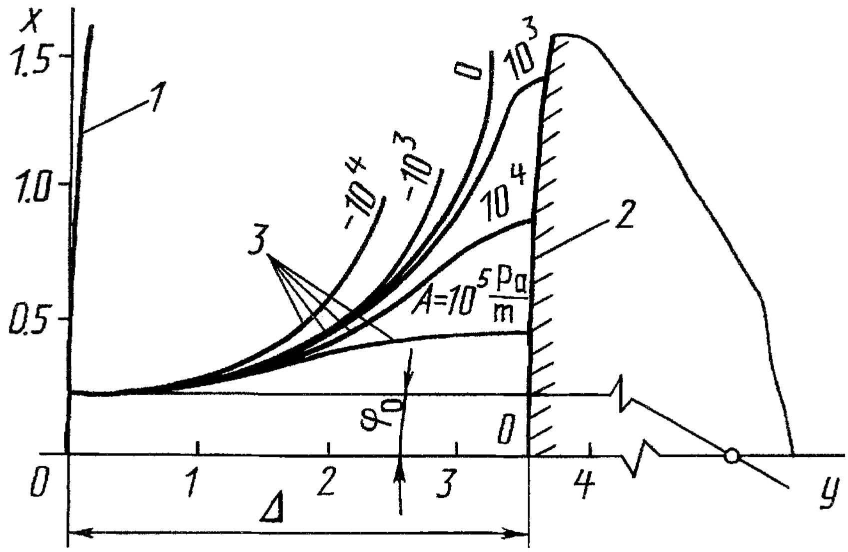

Fig. 2. Pattern of streamlines for a hypersonic air flow around an infinitely long circular cylinder at various gradients A: 1) shock wave; 2) cylindrical surface; 3) streamlines. x, y, mm.

6. The results obtained are used to construct a model of the flow around convex symmetric bodies with suction or injection: it is assumed that the surface of the body in the flow is permeable and a constant pressure gradient A persists in the direction $n-n$ (Fig. 1), and, depending on the direction, simulates suction or injection of material through the surface of the body. Near the stagnation point, the pressure gradient is expressed by

$$\nabla p = \nabla p_N \exp\left[-\frac{y_0\,|\mathrm{A}|}{y\,|\nabla p_N|}\right] + \mathrm{A}\,, \tag{40}$$

where ∇ is the Hamiltonian; y_0 is a reference value of the coordinate y for which the flow parameters are known, for example, the distance Δ from the body surface to the shock wave (Fig. 2); ∇p_N is the pressure gradient for an impermeable body. Model expression (40) is similar to the exact solution of the theory of a boundary layer with suction or injection with the only difference that the denominator of the exponent is determined by friction forces [4]. In fact, Eq. (40) is a superposition of the vectors A and ∇p_N, and at least in the extreme cases it gives the correct result. Indeed, at A = 0, $\Delta p = \nabla p_N$ we have a flow around an impermeable contous; at $|\mathrm{A}| >> |\nabla p_N|$ the flow is completely determined by the amount of suction or injection; finally $\nabla p = \mathrm{A}$, at $y = 0$, as is usually assumed in the theory of boundary layer with suction or injection [4]. Therefore, it can be expected that in the intermediate case $|\nabla p_N| \sim |\mathrm{A}|$ model (40) gives a qualitatively true flow pattern.

For illustration, in Fig. 2 the predicted flow pattern is shown for the case of an infinitely long circular cylinder, $d = 2$ cm, in a hypersonic air flow $M_\infty = 5$). This pattern is obtained within model (40) at $p_0 = 0.1$ MPa and $T_0 = 300$ K with various values and directions of the pressure gradient A (positive and negative values correspond to flow with suction and injection, respectively; the value 0 is the case of an impermeable contour) for the streamline entering at the angle $\varphi_0 = 1^\circ$ As was expected, adopted model (40) gives a physical reasonable pattern of streamlines, predicting their separation in the case of injection and their abutting in the case of suction. It has been assumed in the calculations that in the flow region considered the shock wave formed has the shape of a circle concentric with the cylinder, and Δ was defined as the distance at which the flow velocity is equal to the velocity behind the shock wave at $|\mathrm{A}| = 0$ (Fig. 2). This is permissible if $|\mathrm{A}|\Delta << p_2$, where p_2 is the pressure near the stagnation point in the absence of the disturbing gradient A. The components of the vector ∇p_N were determined using results obtained in Secs. 3 and 5.

Recently a large number of schemes have been suggested for gasdynamic separation of gaseous mixtures and isotopes [5, 6]. In these schemes a constituent of the mass flow of the target component is induced by barodiffusion due to the centrifugal pressure gradient that arises in the curvilinear flow field because of the difference in the molecular weights of the components in the mixture. In particular, this effect is observed in the scheme with a separating probe that is simultaneously used for extraction of the target fraction [5, 6]. In order to estimate this effect, it is necessary to determine the flow field in the vicinity of the separating probe. This problem was considered in [7] within the scope of the Navier-Stokes equations, including diffusion. In that work a solution was obtained as an asymptotic series in the inverse Reynolds numbers in an undisturbed flow with a small amount of the heavy fraction extracted. Within the theory the data of [7] agree satisfactorily with experiments [5, 6]. This problem is simplified substantially if model expression (40) is used to determine the flow field near the separating probe. As one can see from Fig. 2, in this case the effect of separation of the gas mixture by centrifugal forces decreases as the amount of the target component extracted increases, since this is accompanied by a decrease in the curvature of the streamlines. However, this situation is typical of gasdynamic separating units [8].

The results of the present work can be summarized as follows:

1) The class of solutions of the steady-state equations of the flow of an incompressible liquid is determined in the Euler approximation for the case where the pressure field is described by an additive function of the coordinates.

2) In particular cases the class of solutions distinguished contains, for example, potential flows such as liquid waves in a channel of finite depth, a flow inside a right angle, and a hypersonic gas flow around a sphere; among nonpotential flows, this class includes, in particular, a hypersonic flow around a cylinder.

3) A simple model of the flow around convex symmetric bodies with suction or injection is suggested that gives a physically reasonable flow pattern near the body and can be of interest in some problems of physicochemical technology, for example, in developing new gasdynamic methods for separation of isotopes and gas mixtures.

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